# Reduction of non-linear d'Alembert equations to two-dimensional equations

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### Abstract

We study conditions of reduction of the multidimensional wave equation $\Box u = F(u)$ - a system of the d'Alembert and Hamilton equations: $y_\mu y_\mu = r(y,z); y_\mu z_\mu = q(y,z); z_\mu z_\mu = s(y,z); \Box y = R(y,z); \Box z = S(y,z)$. We prove necessary conditions for compatibility of such system of the reduction conditions. Possible types of the reduced equations represent interesting classes of two-dimensional parabolic, hyperbolic and elliptic equations. Ansatzes and methods used for reduction of the d'Alembert ($n$-dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. This seemingly simple and partial problem involves many important aspects in the studies of the PDE. The paper was published in Proceedings of the 4th Workshop "Group Analysis of Differential Equations & Integrable Systems", 2009, p. 243-253.

## 1   Introduction

We study conditions of reduction of the multidimensional wave equation

$$\Box u = F(u), \tag{1}$$
$$\Box \equiv \partial_{x_0}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2, \quad u = u(x_0, x_1, \ldots, x_n)$$

by means of the ansatz with two new independent variables [1, 2]

$$u = \varphi(y, z), \tag{2}$$

where $y$, $z$ are new variables. Henceforth $n$ is the number of independent space variables in the initial d'Alembert equation.

These conditions are a system of the d'Alembert and Hamilton equations:

$$y_\mu y_\mu = r(y,z), \quad y_\mu z_\mu = q(y,z), \quad z_\mu z_\mu = s(y,z), \tag{3}$$
$$\Box y = R(y,z), \quad \Box z = S(y,z).$$

We prove necessary conditions for compatibility of such system of the reduction conditions. This paper is a development of research started jointly with W.I. Fushchych in 1990s [3], and we present some new results and ideas.

Possible types of the reduced equations represent interesting classes of two-dimensional equations - parabolic, hyperbolic and elliptic. Ansatzes and methods used for reduction of the d'Alembert ($n$-dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. This seemingly simple and partial problem involves many important aspects in the studies of the PDE.

Classes of exact solutions of non-linear equations having respective symmetry properties can be constructed by means of symmetry reduction of these equations to equations with smaller number of independent variables or to ordinary differential equations (for the algorithms see the books [4] - [7]).

Reductions and solutions of equation (1) by means of symmetry reduction or ansatzes were considered in numerous papers [8]- [13]. See [14] for a review of results related to reduction of a number of wave equations. In the paper [15] an alternative was proposed for the method of application of ansatzes for equation (1) with a degree nonlinearity.

The method of symmetry reduction does not give exhaustive description of all exact solutions for an equation, so other methods for construction of exact solutions may be expedient.

A so-called "direct method" for search of exact solutions of nonlinear partial differential equations (giving wider classes of solutions than the symmetry reduction) was proposed by P. Clarkson and M. Kruskal [16] (see also [17, 18] and the papers cited therein). It is easy to see that this method for majority of equations results in considerable difficulties as it requires investigation of compatibility and solution of cumbersome reduction conditions of the initial equation. These reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations - e.g. in the situation of the nonlinear wave equations.

The direct method, if applied "completely" (with full solution of compatibility conditions), is exhaustive to some extent - it allows obtaining all reductions of the original equations that may be obtained from $Q$-conditional symmetry (see more comments on symmetry in Section 4).

In this paper we were not able to achieve such complete application of the direct reduction to equation (1) - the presented results are only a step to such application. To do that it is necessary to find a general solution of the reduction conditions.

Direct reduction with utilisation of ansatzes or exhaustive description of conditional symmetries (even $Q$-conditional symmetries) cannot be regarded as algorithmic to the same extent as the standard symmetry reduction. Majority of papers on application of the direct method are devoted to evolution equations or other equations that contain variables of the order not higher than one for at least one of the independent variables, with not more than three independent variables. In such cases solution of the reduction conditions is relatively simple.

We consider general reduction conditions of equation (1) by means of a general ansatz with two new independent variables. We found necessary compatibility conditions for the respective reduction conditions – we developed the conditions found in [3]. We also describe respective possible forms of the reduced equations. Thus we proved that the reduced equations may have only a particular form.

A similar problem was considered by previous authors for an ansatz with one independent variable

$$u = \varphi(y), \tag{4}$$

where $y$ is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\Box u = F(u), \quad u_\mu u_\mu = f(u) \tag{5}$$

in the three-dimensional space was done in [19]. For more detailed review of investigation and solutions of this system see [20, 21].

The compatibility condition of the system (5) for $f(u) = 0$ was found in the paper [22].

Complete investigation of compatibility of overdetermined systems of differential equations with fixed number of independent variables may be done by means of Cartan's algorithm [23], however, it is very difficult for practical application even in the case of three independent variables, and not applicable for arbitrary number of independent variables. For this reason some ad hoc techniques for such cases should be used even for search of necessary compatibility conditions.

It is evident that the d'Alembert–Hamilton system (5) may be reduced by local transformations to the form

$$\Box u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \tag{6}$$

Necessary compatibility conditions of the system (6) for four independent variables were studied in [24] (see also [21]). The necessary compatibility conditions for the system (6) for arbitrary number of independent variables were found in [25]:

**Statement 1**. *For the system* (6) *(n is arbitrary) to be compatible it is necessary that the function F have the following form:*

$$F = \frac{\lambda \partial_u \Phi}{\Phi}, \quad \partial_u^{n+1} \Phi = 0.$$

W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko [20, 26, 27] found a general solution of the system (6) for three space variables (that is four independent variables), as well as necessary and sufficient compatibility conditions for this system [26]:

**Statement 2**. *For the system* (6) *($u = u(x_0, x_1, x_2, x_3)$) to be compatible it is necessary and sufficient that the function F have the following form:*

$$F = \frac{\lambda}{N(u+C)}, \quad N = 0, 1, 2, 3.$$

The results presented in this paper may be regarded as development of the above Statements.

Reduction of equation (1) by means of the ansatz (2) was considered in [27] for a special case (when the second independent variable enters the reduced equation only as a parameter), all respective ansatzes for the case of four independent variables were described, and the respective solutions were found. Some solutions of such type for arbitrary $n$ were also considered in [28].

In [29] reduction of the nonlinear d'Alembert equation by means of ansatz $u = \phi(\omega_1, \omega_2, \omega_3)$ was considered for the case $\Box \omega_1 = 0$, $\omega_{1\mu} \omega_{1\mu} = 0$ (that is $\omega_1$ entered the reduced equation only as a parameter). The respective compatibility conditions were studied and new (non-Lie) exact solutions were found. Note that this case does not include completely the case considered here - the case of the ansatz with two new independent variables.

## 2 Necessary compatibility conditions of the system of the d'Alembert–Hamilton equations for two functions or for a complex-valued function

Reduction of multidimensional equations to two-dimensional ones may be interesting as solutions of two-dimensional partial differential equations, including non-linear ones, may be investigated

more comprehensively than solutions of multidimensional equations, and such two-dimensional equations may have more interesting properties than ordinary differential equations. Two-dimensional reduced equations also may have interesting properties with respect to conditional symmetry.

Substitution of ansatz (2) into the equation (1) leads to the following equation:

$$\varphi_{yy} y_\mu y_\mu + 2\varphi_{yz} z_\mu y_\mu + \varphi_{zz} z_\mu z_\mu + \varphi_y \Box y + \varphi_z \Box z = F(\varphi) \tag{7}$$

$$\left( y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right),$$

whence we get a system of equations:

$$y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \tag{8}$$
$$\Box y = R(y, z), \quad \Box z = S(y, z).$$

System (8) is a reduction condition for the multidimensional wave equation (1) to the two-dimensional equation (7) by means of ansatz (2).

The system of equations (8), depending on the sign of the expression $rs - q^2$, may be reduced by local transformations to one of the following types:

1) elliptic case: $rs - q^2 > 0$, $v = v(y, z)$ is a complex–valued function,

$$\Box v = V(v, v^*), \quad \Box v^* = V^*(v, v^*),$$
$$v_\mu^* v_\mu = h(v, v^*), \quad v_\mu v_\mu = 0, \quad v_\mu^* v_\mu^* = 0 \tag{9}$$

(the reduced equation is of the elliptic type);

2) hyperbolic case: $rs - q^2 < 0$, $v = v(y, z)$, $w = w(y, z)$ are real functions,

$$\Box v = V(v, w), \quad \Box w = W(v, w),$$
$$w_\mu w_\mu = h(v, w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0 \tag{10}$$

(the reduced equation is of the hyperbolic type);

3) parabolic case: $rs - q^2 = 0$, $r^2 + s^2 + q^2 \neq 0$, $v(y, z)$, $w(y, z)$ are real functions,

$$\Box v = V(v, w), \quad \Box w = W(v, w),$$
$$v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda = \pm 1), \quad w_\mu w_\mu = 0 \tag{11}$$

(if $W \neq 0$, then the reduced equation is of the parabolic type);

4) first-order equations: $(r = s = q = 0)$, $y \to v$, $z \to w$

$$v_\mu v_\mu = w_\mu w_\mu = v_\mu w_\mu = 0,$$
$$\Box v = V(v, w), \quad \Box w = W(v, w). \tag{12}$$

Let us formulate necessary compatibility conditions for the systems (9)–(12).

**Theorem 1**. *System* (9) *is compatible if and only if*

$$V = \frac{h(v, v^*)\partial_{v^*}\Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

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where $\Phi$ is an arbitrary function for which the following condition is satisfied

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

The function $h$ may be represented in the form $h = \frac{1}{R_{vv^*}}$, where $R$ is an arbitrary sufficiently smooth function, $R_v$, $R_{v^*}$ are partial derivatives by the respective variables.

Then the function $\Phi$ may be represented in the form $\Phi = \sum\limits_{k=0}^{n+1} f_k(v)R_v^k$, where $f_k(v)$ are arbitrary functions, and

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v)R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v)R_v^k}.$$

The respective reduced equation will have the form

$$h(v, v^*)\left(\phi_{vv^*} + \phi_v\frac{\partial_{v^*}\Phi}{\Phi} + \phi_{v^*}\frac{\partial_v\Phi^*}{\Phi^*}\right) = F(\phi). \tag{13}$$

The equation (13) may also be rewritten as an equation with two real independent variables $(v = \omega + \theta, \ v^* = \omega - \theta)$:

$$2\widetilde{h}(\omega, \theta)(\phi_{\omega\omega} + \phi_{\theta\theta}) + \Omega(\omega, \theta)\phi_\omega + \Theta(\omega, \theta)\phi_\theta = F(\phi). \tag{14}$$

We will not adduce here cumbersome expressions for $\Omega$, $\Theta$ that may be found from (13).

**Theorem 2**. *System* (10) *is compatible if and only if*

$$V = \frac{h(v, w)\partial_w\Phi}{\Phi}, \quad W = \frac{h(v, w)\partial_v\Psi}{\Psi},$$

*where the functions $\Phi$, $\Psi$ are arbitrary functions for which the following conditions are satisfied*

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

The function $h$ may be presented in the form $h = \frac{1}{R_{vw}}$, where $R$ is an arbitrary sufficiently smooth function, $R_v$, $R_w$ are partial derivatives by the respective variables. Then the functions $\Phi$, $\Psi$ may be represented in the form

$$\Phi = \sum_{k=0}^{n+1} f_k(v)R_v^k, \quad \Psi = \sum_{k=0}^{n+1} g_k(w)R_w^k,$$

where $f_k(v)$, $g_k(w)$ are arbitrary functions,

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v)R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v)R_v^k}, \quad W = \frac{\sum\limits_{k=1}^{n+1} k g_k(w)R_w^k}{\sum\limits_{k=0}^{n+1} g_k(w)R_w^k}.$$

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The respective reduced equation will have the form

$$h(v, w) \left( \phi_{vw} + \phi_v \frac{\partial_w \Phi}{\Phi} + \phi_w \frac{\partial_v \Psi}{\Psi} \right) = F(\phi). \tag{15}$$

**Theorem 3**. *System* (11) *is compatible if and only if*

$$V = \frac{\lambda \partial_v \Phi}{\Phi}, \quad \partial_v^{n+1} \Phi = 0, \quad W \equiv 0.$$

Equation (1) by means of ansatz (2) cannot be reduced to a parabolic equation – in this case one of the variables will enter the reduced ordinary differential equation of the first order as a parameter.

Compatibility and solutions of such system for $n = 3$ were considered in [27]; for this case necessary and sufficient compatibility conditions, as well as a general solution, were found.

System (12) is compatible only in the case if $V = W \equiv 0$, that is the reduced equation may be only an algebraic equation $F(u)=0$. Thus we cannot reduce equation (1) by means of ansatz (2) to a first-order equation.

Proof of these theorems is done by means of utilisation of lemmas similar to those adduced in [24, 25], and of the well-known Hamilton–Cayley theorem, in accordance to which a matrix is a root of its characteristic polynomial.

We will present an outline of proof of Theorem 2 for the hyperbolic case. For other cases the proof is similar.

We will operate with matrices of dimension $(n + 1) \times (n + 1)$ of the second variable of the functions $v$ and $w$:

$$\hat{V} = \{v_{\mu\nu}\}, \quad \hat{W} = \{w_{\mu\nu}\}.$$

With respect to operations with these matrices we utilise summation arrangements customary for the Minkowsky space: $v_0 = i\partial_{x_0}$, $v_a = -i\partial_{x_a} (a = 1, \ldots, n)$, $v_\mu v_\mu = v_0^2 - v_1^2 - \cdots - v_n^2$.

**Lemma 1**. *If the functions $v$ and $w$ are solutions of the system* (10), *then the following relations are satisfied for them for any $k$:*

$$\mathrm{tr}\hat{V} = \frac{(-1)^k}{(k-1)!}(h(v, w)\partial_w)^{k+1}V(v, w),$$

$$\mathrm{tr}\hat{W} = \frac{(-1)^k}{(k-1)!}(h(v, w)\partial_v)^{k+1}W(v, w).$$

**Lemma 2**. *If the functions $v$ and $w$ are solutions of the system* (10), *then* $\det \hat{V} = 0$, $\det \hat{W} = 0$.

**Lemma 3**. *Let $M_k(\hat{V})$ be the sum of principal minors of the order $k$ for the matrix $\hat{V}$. If the functions $v$ and $w$ are solutions of the system* (10), *then the following relations are satisfied for them for any $k$:*

$$M_k(\hat{V}) = \frac{(h(v, w)\partial_w)^k \Phi}{k!\Phi}, \quad M_k(\hat{W}) = \frac{(h(v, w)\partial_v)^k \Psi}{k!\Psi},$$

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*where the functions $\Phi$, $\Psi$ satisfy the following conditions*

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

These lemmas may be proved with the method of mathematical induction similarly to [25] with utilisation of the Hamilton–Cayley theorem ($E$ is a unit matrix of the dimension $(n+1) \times (n+1)$).

$$\sum_{k=0}^{n-1}(-1)^k M_k \hat{V}^{n-k} + (-1)^n E \det \hat{V} = 0.$$

It is evident that the statement of Theorem 2 is a direct consequence of Lemma 3 for $k = 1$.

**Note 1**. Equation (9) may be rewritten for a pair of real functions $\omega = \operatorname{Re} v$, $\theta = \operatorname{Im} v$. Though in this case necessary the respective compatibility conditions would have extremely cumbersome form.

**Note 2**. Transition from (8) to (9)–(12) is convenient only from the point of view of investigation of compatibility. The sign of the expression $rs - q^2$ may change for various $y$, $z$, and the transition is being considered only within the region where this sign is constant.

## 3 Examples of solutions of the system of d'Alembert–Hamilton equations

Let us adduce explicit solutions of systems of the type (8) and the respective reduced equations. Parameters $a_\mu$, $b_\mu$, $c_\mu$, $d_\mu$ ($\mu = 0, 1, 2, 3$) satisfy the conditions:

$$-a^2 = b^2 = c^2 = d^2 = -1 \quad (a^2 \equiv a_0^2 - a_1^2 - \cdots - a_3^2),$$
$$ab = ac = ad = bc = bd = cd = 0;$$

$y$, $z$ are functions of $x_0$, $x_1$, $x_2$, $x_3$.

    1)   $y = ax, \quad z = dx, \quad \varphi_{yy} - \varphi_{zz} = F(\varphi);$

    2)   $y = ax, \quad z = \left((bx)^2 + (cx)^2 + (dx)^2\right)^{1/2},$

        $\varphi_{yy} - \varphi_{zz} - \dfrac{2}{z}\varphi_z = F(\varphi);$

In this case the reduced equation is a so-called radial wave equation, the symmetry and solutions of which were investigated in [38, 39].

    3)   $y = bx + \Phi(ax + dx), \quad z = cx, \quad -\varphi_{zz} - \varphi_{yy} = F(\varphi);$

    4)   $y = \left((bx)^2 + (cx^2)\right)^{1/2}, \quad z = ax + dx, \quad -\varphi_{yy} - \dfrac{1}{y}\varphi_y = F(\varphi).$

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# 4 Symmetry aspects

Solutions obtained by the direct reduction are related to symmetry properties of the equation – $Q$-conditional symmetry of this equation [6, 30, 31] (symmetries of such type are also called non-classical or non-Lie symmetries [16, 32, 33]). For more theoretical background of conditional symmetry and examples see also [14], [34], [35].

Conditional symmetry and solutions of various non-linear two-dimensional wave equations that may be regarded as reduced equations for equation (1) were considered in [36]-[40]. It is also possible to see from these papers that symmetry of the two-dimensional reduced equations is often wider than symmetry of the initial equation, that is the reduction to two-dimensional equations allows to find new non-Lie solutions and hidden symmetries of the initial equation (see e.g. [41]).

The Hamilton equation may also be considered, irrespective of the reduction problem, as an additional condition for the d'Alembert equation that allows extending the symmetry of this equation. The symmetry of the system

$$\Box u = F(u), \quad u_\mu u_\mu = 0$$

was described in [42]. In [25] a conformal symmetry of the system (5) was found that is a new conditional symmetry for the d'Alembert equation. Conditional symmetries of this system were also described in [27, 29].

# 5 Conclusions

The results of investigation of compatibility and solutions of the systems (9)–(12) may be utilised for investigation and search of solutions also of other Poincaré–invariant wave equations, beside the d'Alembert equation, e.g. Dirac equation, Maxwell equations and equations for the vector potential.

Thus, in the present paper we found

1) necessary compatibility conditions for the system of the d'Alembert–Hamilton equations for two dependent functions, that is reduction conditions of the non-linear multidimensional d'Alembert equation by means of ansatz (2) to a two-dimensional equation; such compatibility conditions for equations of arbitrary dimensions cannot be found by means of the standard procedure.

2) possible types of the two-dimensional reduced equations that may be obtained from equation (1) by means of ansatz (2).

The found reduction conditions and types of ansatzes may be also used for arbitrary Poincaré–invariant multidimensional equation. In [43] the general form of the scalar Poincaré–invariant multidimensional equations were described; it is easy to prove that by means of ansatz (2) it is possible to reduce all these equations to PDE in two independent variables.

# 6 Further Research

1. Study of Lie and conditional symmetry of the system of the reduction conditions (symmetry of the system of the d'Alembert equations for the complex function was investigated

in [44]).

2. Investigation of Lie and conditional symmetry of the reduced equations. Finding exact solutions of the reduced equations.

3. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.

4. Group classification of the reduced equations.

5. Finding of sufficient compatibility conditions and of a general solution of the compatibility conditions for lower dimensions ($n = 2, 3$).

6. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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